1. We assume the agent returns home after each visit, so the same city may be visited several times in a row. The total number of ways to visit 3 cities 3 times each is $\binom{3+3+3}{3,3,3} = \binom{9}{3,3,3} = \frac{9!}{3!3!3!} = 1680$. If, in addition, the agent does start and finish in the same city, the there are $\binom{3}{1} = 3$ ways to pick this first/last city. This city is visited first, last, and 1 more time out of remaining 7 visits. Thus, the number of forbidden itineraries is $3\binom{7}{1,3,3} = 3\frac{7!}{1!3!3!} = 3 \cdot 140 = 420$, so the number of allowed itineraries is $\binom{9}{3,3,3} - 3\binom{7}{1,3,3} = 1680 - 420 = 1260$.

2. Note that the same colors in different order make a different flag, so the total number of different flags is the number 3-permutations of [4], i.e. $P(4,3) = \frac{4!}{(4-3)!} = 24$. Therefore, by the pigeonhole principle, at least one flag must occur $\lceil \frac{49}{24} \rceil = \lceil 2\frac{1}{24} \rceil = 3$ times.

3. The number of ways to choose 2 different courses out of 13 is $\binom{13}{2} = \frac{13 \cdot 12}{2 \cdot 1} = 78$. The professor taught for $40 \cdot 2 = 80$ semesters. Since $80 > 78$, by the pigeonhole principle, there were at least 2 semesters when the professor had identical teaching programs.

4. *Solution 1.* The total number of ways to work 5 days out of 7 is $\binom{7}{5} = 21$. The non-preferred schedules are those that include both Saturday and Sunday, i.e. consist of 3 weekdays out of 5 and 2 weekend days out of 2. Thus, their number is $\binom{5}{3}\binom{2}{2} = 10 \cdot 1 = 10$. Therefore, the number of preferred ways is $\binom{7}{5} - \binom{5}{3}\binom{2}{2} = 21 - 10 = 11$.

   Solution 2. The cashier does not want to work both days of the weekend, hence he either wants to work 5 out of 5 weekdays and 0 out of 2 weekend days, or 4 out of 5 weekdays and 1 out of 2 weekend days, for the total of $\binom{5}{5}\binom{2}{0} + \binom{5}{4}\binom{2}{1} = 1 \cdot 1 + 5 \cdot 2 = 11$ ways.

5. The probability that any given sorter does not show up for work is 1/2, so the probability that all 3 sorters don't show up is $(1/2)^3$. Therefore, the probability that at least one sorter does show up is $1 - (1/2)^3 = 7/8$. Similarly, the probability that at least one out of two packagers shows up is $1 - (1/2)^2 = 3/4$. Hence, the probability that at least 1 sorter and at least one packager show up is $(1 - (1/2)^3)(1 - (1/2)^2) = (7/8)(3/4) = 21/32$.

6. The total number of possible digits in each of 6 positions is 10, so the total number of possible license plates is 10^6. Let us now find the number of license plates that satisfy the witness' description. (Recall that the criminal's license plate had 4 different digits: 1 occurred 3 times, and the remaining 3 occurred 1 time each.) We can do that in either of the two following ways.

   Approach 1. Choose the 3 positions to be occupied by the same digit in $\binom{6}{3} = 20$ ways. Then choose an ordered sequence of 4 digits out of 10 in $P(10,4) = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$ ways (in other words, it matters in which order we choose these digits). Then there is only 1 way to put the first chosen digit in the 3 chosen positions, then put the 3 remaining digits in the 3 remaining positions, left to right in the order these digits were chosen (for example, positions $1, 2, 5$ and digits $4, 3, 7, 5$ give us the license plate 443745). Thus, the number of suspect license plates is $\binom{6}{3}P(10,4) = 20 \cdot 5040 = 100800$.

   Approach 2. Choose the 4 different digits a, b, c, d to occur in the license plate in $\binom{10}{4} = 210$ ways. Among them, choose 1 special digit (the one that occurs 3 times, say, d) in $\binom{4}{1} = 4$ ways. Now permute the set *abcddd* in any of $\binom{6}{1,1,1,3} = \frac{6!}{1!1!1!3!} = 120$ ways. Thus, the number of suspect license plates is $\binom{10}{4}\binom{4}{1}\binom{6}{1,1,1,3} = 210 \cdot 4 \cdot 120 = 100800$.

   Thus, the probability that a given license plate matches witness' description is $100800/10^6 = 0.1008$, i.e. 10.08%, so $100 - 10.08 = 89.92 < 90$ percent of license plates are excluded, and hence, the police officer's estimate was incorrect (although pretty close).

7. (a) **(A Goofy Judge)** Here is the table of probabilities of conviction and acquittal for each judge. Recall that $Prob(Convict) + Prob(Acquit) = 1$ for each judge.

	Goofy judge	Serious judge 1	Serious judge 2
Convict	1/2	p	p
Acquit	1/2	$1-p$	$1-p$

If the prisoner is convicted, then either all 3 judges voted to convict, or 2 voted to convict and 1 voted to acquit. Therefore, the probability of conviction is

$$\frac{1}{2}pp + \frac{1}{2}p(1-p) + \frac{1}{2}(1-p)p + \frac{1}{2}pp = \frac{p^2}{2} + \frac{p-p^2}{2} + \frac{p-p^2}{2} + \frac{p^2}{2} = p,$$

the same as if the prisoner were tried by a single serious judge. Thus, the prisoner is equally well off either way.

(b) **(Serious Judges)** This is a similar problem except each judge convicts with probability p and acquits with probability $1-p$. If the prisoner is convicted, then either all 3 judges voted to convict, or 2 out of 3 voted to convict and 1 out of 3 voted to acquit. Therefore, the probability of conviction is

$$p^3 + \binom{3}{2}p^2(1-p) = p^3 + 3(p^2 - p^3) = 3p^2 - 2p^3.$$

How does this compare with p? Well, consider

$$p - (3p^2 - 2p^3) = p - 3p^2 + 2p^3 = p(1 - 3p + 2p^2) = p(1-p)(1-2p).$$

Thus, the equality is achieved when $p = 0, 1, \frac{1}{2}$ (i.e. if all judges always acquit, always convict, or are all goofy). For the remaining values of p, we know that $0 < p < 1$, so $p > 0$ and $1 - p > 0$, so sign of the above difference is the same as the sign of $1 - 2p$, i.e. positive if $0 < p < 1/2$ and negative if $1/2 < p < 1$.

Thus, the prisoner is equally well of either way if $p = 0, \frac{1}{2}, 1$; better off with 3 judges if $0 < p < 1/2$; and better off with 1 judge if $1/2 < p < 1$.