

1. Proof by induction.

*Base.* The statement is true for  $n = 1$  since  $1^3 = 1 = 1^2 \cdot (1 + 1)^2 / 4$ .

*Step.* Assume the statement is true for some integers  $n \geq 1$ . Then  $1^3 + 2^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$ , so

$$\begin{aligned} 1^3 + 2^3 + \dots + n^3 + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2(n^2 + 4(n+1))}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4} = \frac{(n+1)^2((n+1)+1)^2}{4}, \end{aligned}$$

hence our statement is true for  $n + 1$  as well. Therefore,  $1^3 + 2^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$  for all integer  $n \geq 1$ .

2. Proof by induction.

*Base.* Let  $n = 0$ , then  $a_0 = 1 = 2 \cdot 1 - 1 = 2 \cdot 3^0 - 1$ .

*Step.* Assume the statement is true for some integer  $n \geq 0$ , then  $a_n = 2 \cdot 3^n - 1$ , so  $a_{n+1} = 3a_n + 2 = 3(2 \cdot 3^n - 1) + 2 = 2 \cdot 3^{n+1} - 1$ . Hence, our statement is true for  $n + 1$  as well.

Therefore, by induction,  $a_n = 2 \cdot 3^n - 1$  for all integer  $n \geq 0$ .

3. Let  $a_m$  be the number of squares after  $m$  squares were cut. Then  $a_0 = 1$  since we started with 1 square. Every time we make a cut, we get 4 small squares in place of 1 large square, thus,  $a_{m+1} = a_m + 4 - 1 = a_m + 3$ . Let us prove by induction that  $a_m = 3m + 1$ .

*Base.*  $a_0 = 1 = 3 \cdot 0 + 1$ .

*Step.* Assume  $a_m = 3m + 1$  for some integer  $m \geq 0$ . Then  $a_{m+1} = a_m + 3 = 3m + 1 + 3 = 3(m + 1) + 1$ , so our claim is true for  $m + 1$  as well.

Therefore, by induction,  $a_m = 3m + 1$  for all integers  $m \geq 0$ , so  $a_m \equiv 1 \pmod{3}$ .

4. Proof by induction.

*Base.* The statement is true for  $n = 0$ , since  $a_0 = 8^0 - 14 \cdot 0 + 27 = 1 - 0 + 27 = 28$  is divisible by 7.

*Step.* Assume that  $a_n$  is divisible by 7 for some integer  $n \geq 0$ . To prove  $a_{n+1}$  is also divisible by 7, we only need to prove that  $a_{n+1} - a_n$  is divisible by 7. But

$$a_{n+1} - a_n = (8^{n+1} - 14(n+1) + 27) - (8^n - 14n + 27) = 8^n(8 - 1) - 14 = 8^n \cdot 7 - 14 = 7(8^n - 2)$$

is clearly divisible by 7. Therefore,  $a_{n+1} = a_n + (a_{n+1} - a_n)$  is also divisible by 7.

Thus, by induction,  $a_n = 8^n - 14n + 27$  is divisible by 7 for any  $n \geq 0$ .

5. We will prove by induction that  $b_n = 3^n/n^4 > 1$  for all integers  $n \geq 8$ .

*Base.* Our statement is true for  $n = 8$  since  $b_8 = 3^8/8^4 = 9^4/8^4 > 1$ .

*Step.* Assume that  $b_n > 1$  for some  $n \geq 8$ . To prove  $b_{n+1} > 1$ , we only need to show that  $b_{n+1}/b_n > 1$ . But

$$\frac{b_{n+1}}{b_n} = \frac{3^{n+1}}{(n+1)^4} = \frac{3n^4}{(n+1)^4},$$

and

$$\frac{3n^4}{(n+1)^4} > 1 \iff \frac{(n+1)^4}{n^4} < 3 \iff \left(1 + \frac{1}{n}\right)^4 < 3 \iff 1 + \frac{1}{n} < \sqrt[4]{3} \iff n > \frac{1}{\sqrt[4]{3} - 1}.$$

Since  $8 > 1/(\sqrt[4]{3} - 1)$ , it follows that  $b_{n+1}/b_n > 1$  for any  $n \geq 8$ . Therefore,  $b_{n+1} = b_n(b_{n+1}/b_n) > b_n > 1$ .

Thus, by induction,  $b_n > 1$  for all  $n \geq 8$ .

6. Let  $a_0, a_1, \dots, a_n$  be the digits of a positive integer  $m$ , from right to left. Prove that  $m$  is divisible by 11 if and only if  $a_0 - a_1 + a_2 - a_3 + \dots + (-1)^n a_n$  (the *alternating sum* of the digits) is divisible by 11.

We will prove by induction on  $n$  that

$$m = a_n a_{n-1} \dots a_1 a_0 \equiv a_0 - a_1 + a_2 - a_3 + \dots + (-1)^n a_n \pmod{11}.$$

*Base.* The above claim is true for  $n = 0$ , since we simply get  $a_0 \equiv a_0 \pmod{11}$  in that case.

*Step.* Assume our claim is true for some integer  $n \geq 0$ . Then, for any  $(n+1)$ -digit number  $m = a_n a_{n-1} \dots a_1 a_0$ , we have  $m \equiv a_0 - a_1 + a_2 - a_3 + \dots + (-1)^n a_n \pmod{11}$ . Consider any  $(n+2)$ -digit number  $M = b_{n+1} b_n b_{n-1} \dots b_1 b_0$ . The number  $b_{n+1} b_n b_{n-1} \dots b_1$  has  $n+1$  digits, hence our induction hypothesis can be applied to it, i.e.

$$b_{n+1} b_n \dots b_2 b_1 \equiv b_1 - b_2 + b_3 - b_4 + \dots + (-1)^n b_{n+1} \pmod{11}.$$

But  $M = b_{n+1} b_n b_{n-1} \dots b_1 b_0 = b_{n+1} b_n b_{n-1} \dots b_1 \cdot 10 + b_0$ , and  $10 \equiv -1 \pmod{11}$ , so

$$\begin{aligned} M &\equiv 10(b_1 - b_2 + b_3 - b_4 + \dots + (-1)^n b_{n+1}) + b_0 \pmod{11} \\ &\equiv (-1)(b_1 - b_2 + b_3 - b_4 + \dots + (-1)^n b_{n+1}) + b_0 \pmod{11} \\ &= b_0 - b_1 + b_2 - b_3 + b_4 - \dots + (-1)^{n+1} b_{n+1} \pmod{11}, \end{aligned}$$

as claimed. Therefore, our statement is true for  $n+1$  as well.

Thus, our claim is true by induction on  $n$ , i.e. any integer is divisible by 11 with the same remainder as the alternating sum of its digits. A simple corollary of that is that an integer is divisible by 11 (with remainder 0) if and only if the alternating sum of its digits is also divisible by 11.