Math 304 **Homework 2** **Due 9/26/2002**

1. Prove that $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4} = (1 + 2 + \cdots + n)^2$.

2. Let $a_0 = 1$, and let $a_{n+1} = 3a_n + 2$ for any nonnegative integer n. Prove that $a_n = 2 \cdot 3^n - 1$ for all integer $n \geq 0$.

3. We cut a square into four smaller squares, then we cut some of the obtained small squares into four smaller squares, and so on. Prove that at any given point during this operation the number of all squares we have is of the form $3m + 1$ for some integer m ("1 modulo 3" in mathspeak, denoted "$1 \mod 3$").

4. Prove that if n is a nonnegative integer, then $a_n = 8^n - 14n + 27$ is divisible by 7.

5. Prove that $3^n > n^4$ if $n \geq 8$.

6. Let $a_0, a_1, \ldots, a_n$ be the digits of a positive integer m, from right to left. Prove that m is divisible by 11 if and only if $a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n$ (the *alternating sum* of the digits) is divisible by 11.

Hints:

3. How many more squares do we get by cutting any one square?

4. Consider a_0, and $a_{n+1} - a_n$ for $n \geq 0$.

5. Let $b_n = 3^n/n^4$ and prove $b_n > 1$ for $n \geq 8$. To do this, prove $b_8 > 1$, and $b_{n+1}/b_n > 1$ for $n \geq 8$.

6. Note that 10 is -1 modulo 11 (i.e. 10 is divisible by 11 with remainder -1, denoted "$10 \equiv -1 \mod 11$"). Also, $m = a_n a_{n-1} \ldots a_1 a_0 = a_n a_{n-1} \ldots a_1 \cdot 10 + a_0$ (here, $a_n a_{n-1} \ldots a_1 a_0$ is not a product but simply a string of digits).