1. (a) (Problem 8.1.1.) Use the Pigeonhole Principle.

   i. There are 26 possible initials (pigeonholes). To make sure at least 2 people have the same first initial, we need $26 + 1 = 27$ people.

   ii. There are at most 366 days in a year. To make sure at least 2 people have a birthday on the same days of the year, we need $366 + 1 = 367$ people.

   (b) (Problem 8.1.2.) Use the Generalized Pigeonhole Principle (Theorem 8.2). If we have m pigeons and k pigeonholes, then we need $\lfloor \frac{m-1}{k} \rfloor = 3 - 1 = 2$ here, i.e. $m = 2k + 1$

   i. $k = 26$, so $m = 2 \cdot 26 + 1 = 53$.

   ii. $k = 366$, so $m = 2 \cdot 366 + 1 = 733$.

   (c) (Problem 8.1.12.) Let s_i be the total number of visits the social worker makes on days 1 through i, where $i = 1, 2, \dots, 77$. Then $s_{77} = 132$, $s_1 \geq 1$, and $s_i > s_{i-1}$ for any $i = 2, \dots, 77$, since he makes at least 1 visit on each day. Thus, we have

   $$1 \leq s_1 < s_2 < \cdots < s_{77} = 132,$$

   so

   $$22 \leq s_1 + 21 < s_2 + 21 < \cdots < s_{77} + 21 = 153.$$

   Thus, the set $\{s_1, \dots, s_{77}, s_1 + 21, \dots, s_{77} + 21\}$ has $77 + 77 = 154$ integers with values from 1 to 153. by the Pigeonhole Principle, two of them must be equal. Since $s_i \neq s_j$ and $s_i + 21 \neq s_j + 21$ if $i \neq j$, we must have $s_j = s_i + 21$ for some $1 \leq i < j \leq 77$. But then $21 = s_j - s_i$ is the number of visits made on days $i + 1$ through j. QED.

   (d) (Problem 8.1.17.) Let a_i be the number of hours the employee worked on day $i = 1, 2, \dots, 10$. Then $a_1 + a_2 + \cdots + a_{10} = 81$. Assume that he worked less than 17 hours (i.e. at most 16 hours) on each pair of consecutive days. Then $a_i + a_{i+1} \leq 16$ for each $i = 1, 2, \dots, 9$. In particular, $a_1 + a_2 \leq 16$,$a_3 + a_4 \leq 16$,$a_5 + a_6 \leq 16$,$a_7 + a_8 \leq 16$,$a_9 + a_{10} \leq 16$. adding up these 5 inequalities, we get $81 = a_1 + \cdots + a_{10} \leq 16 \cdot 5 = 80$; a contradiction. Therefore, the employee worked at least 17 hours on some pair of consecutive days.

   (e) (Problem 8.1.24.) Note that every integer m may be written as $m = p \cdot 2^q$, where p is an odd integer, and q is a nonnegative integer. since there are n odd integers from 1 to $2n$, and we choose $n + 1$ integers from $\{1, \dots, 2n\}$, at least two of the chosen integers, say, $m_1 < m_2$, will have the same odd factor p in the above factorization, say, $m = p \cdot 2^{q_1}$ and $m_2 = p \cdot 2^{q_2}$. But then $q_2 > q_1$, so $\frac{m_2}{m_1} = 2^{q_2 - q_1}$ is an integer, so $m_1 | m_2$ (m_1 divides m_2).

   (f) (Problem 8.1.25.) Each out of n people may have from 0 up to $n - 1$ acquaintances. Arrange the numbers of acquaintances in nondecreasing order, i.e. $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq n - 1$. But we can't have $a_1 = 0$ and $a_n = n - 1$ simultaneously, since that would mean one person doesn't know anyone in the group and another person knows everyone in the group, which is impossible, since these two people cannot be either acquainted or not acquainted. Thus, $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq n - 1$ or $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq n - 2$. In either case, there are n integers $a_1, \dots, a_n$ which may take up to $n - 1$ values. Hence by the pigeonhole principle, two of them must be equal. Therefore, some 2 people in the group will have the same number of acquaintances in the group.

   (g) (Problem 8.1.26.) Consider partial sums $s_i = a_1 + a_2 + \cdots + a_i$, where $1 \leq i \leq p$, and their pairwise differences $s_j - s_i = a_{i+1} + \cdots + a_j$, $i < j$. If the sums s_i, $i = 1, 2, \dots, p$ are not divisible by p, then their remainders modulo p (i.e. from division by p) must be in the set $\{1, 2, \dots, p - 1\}$. Since there are p partial sums, by the Pigeonhole Principle, two of them, say, s_i and s_j $(i < j)$ must have the same remainder modulo p. But then $s_j - s_i$ has remainder 0 modulo p, i.e. $a_{i+1} + \cdots + a_j = s_j - s_i$ is divisible by p (we write $p | (s_j - s_i)$ to denote this). QED.

2. There are $60 \cdot 24 = 1440$ minutes in a day. Since there are 1500 takeoffs per day, two of them must fall within the same minute, hence they are within a minute from each other.

3. Subdivide the square into 9 squares of side $1/3$. Since we have 10 points, two of them must be in the same little square, so the distance between them must be at most the length of the diagonal of that little square, which is $\sqrt{2}/3$.

4. Let $a_i = \underbrace{1970\,1970\ldots1970}_{i}$ (1970 repeated i times), where $i = 1, 2, \ldots, 1971$. Assume no a_i is divisible by 1971. We have 1971 a_i's and only 1970 possible remainders modulo 1971 $(1, 2, \ldots, 1970$; recall that we assumed remainder 0 does not occur among a_i's), so by the Pigeonhole Principle, two of the a_i's, say a_m and a_n $(m < n)$, should have the same remainder modulo 1971. Then

$$a_n - a_m = \underbrace{1970\ldots1970}_{n-m\text{ groups}}\underbrace{0000\ldots0000}_{m\text{ groups}} = a_{n-m} \cdot 10^{4m}$$

is divisible by 1971. Any divisor of 10^{4m} is a product of some power of 5 and some power of 2, but $5 \not| 1971$ and $2 \not| 1971$, therefore 10^{4m} and 1971 have no common divisors except 1. This means that a_{n-m} is divisible by 1971, which contradicts our assumption since $1 \le n-m \le 1970$. Therefore, one of the a_i's is divisible by 1971.

5. Since any decreasing subsequence has at most 3 terms, we conclude that no decreasing subsequence has $3 + 1 = 4$ terms. We have $13 = 3 \cdot 4 + 1$, so by the Erdös-Szekeres theorem, our sequence must have an increasing subsequence of length at least $4 + 1 = 5$. Is this minimum actually achieved in some case? Yes. The sequence $3, 2, 1, 6, 5, 4, 9, 8, 7, 12, 11, 10, 13$ has no decreasing subsequence with $3 + 1 = 4$ terms, and its longest increasing subsequence is of length 5. Hence, the minimum size of the longest increasing subsequence in our case is 5.