Math 304 **Final Exam** **Due 12/18/2002**

This exam is due Wednesday, December 18, in my office, Carver 456, by 5pm. You may consult the text for this course, the handouts, the homework solutions, your notes taken in lecture and your homework. Do not use any other books or papers or materials from a library or consult with any person other than myself. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the above rules. Remember to SHOW ALL WORK.

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1. Prove that
$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$
Hint: Note that $\binom{n}{k}^2 = \binom{n}{k}\binom{n}{n-k}$.

2. Find the ordinary generating function for the *harmonic* numbers
$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$
Hint: Recognize $\{H_n\}$ as the convolution of two simpler sequences. You may want to use the handout with the list of useful power series to find their generating functions.

3. Use the method of characteristic roots to solve the following recurrence relations:
   (a) $a_n = 3a_{n-1} - 2a_{n-2}$; $a_0 = 0$, $a_1 = 1$.
   (b) $a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$; $a_0 = 1$, $a_1 = 0$, $a_2 = 1$.

4. Find an explicit formula for $\{g_n\}_{n=0}^{\infty}$ given by
$$g_n = -ng_{n-1} + \sum_k \binom{n}{k} g_k g_{n-k}, \quad n \geq 2; \qquad g_0 = 0,\ g_1 = 1.$$
Hint: You may want to use an exponential generating function and recall the exponential convolution. Alternatively, guess the exact formula for $f(n)$, then prove it by induction.

5. Let $a_n = F_{2n}$ for all $n \geq 0$, i.e. $\{a_n\}$ is the sequence of Fibonacci numbers with even indices. Find the recurrence relation the sequence $\{a_n\}$ satisfies and its initial values. *Hint:* Use the fact that $F_k = F_{k-1} + F_{k-2}$ ($k \geq 2$) three times for various k.

6. (a) Let $g(n,k)$ be the number of k-subsets of $[n]$ that contain no two consecutive elements (where $k = 0, 1, 2, \ldots, n$). Prove that

$$g(n,k) = \binom{n+1-k}{k}.$$

Hint: If $A = \{i_1 < i_2 < \cdots < i_k\}$ is a subset of $[n]$ counted by $g(n,k)$, then $i_1 + 1 \notin A$, $i_2 + 1 \notin A$, etc., $i_{k-1} + 1 \notin A$.

(b) By comparing the results of part (a) and Problem 6, Midterm 2, deduce the identity

$$F_n = \sum_k \binom{n-k}{k} = \sum_k \binom{k}{n-k}, \quad n \geq 0,$$

where F_n is the nth Fibonacci number.

7. *(Extra credit)* Every positive integer can be written uniquely as the product of powers of primes, $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$ (e.g. $100 = 2^2 \cdot 5^2$, $14 = 2^1 \cdot 7^1$, $18 = 2^1 \cdot 3^2$, $30 = 2^1 \cdot 3^1 \cdot 5^1$). Define the $\mu(n)$, the *Möbius function* of n, as follows:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } e_1 = e_2 = \cdots = e_r = 1, \\ 0 & \text{otherwise (i.e. if one or more } e_i\text{'s are greater than 1).} \end{cases}$$

(E.g. $\mu(100) = 0$, $\mu(18) = 0$, $\mu(14) = (-1)^2 = 1$, $\mu(30) = (-1)^3 = -1$). Prove that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases}$$

where the sum is over all positive integers d that divide n (including 1 and n itself). *Hint:* For which divisors d of n do we have $\mu(d) \neq 0$? How many such d's are there that have exactly k distinct prime divisors? What is $\mu(d)$ for those d? Now sum the nonzero $\mu(d)$'s for each $k = 0, 1, 2, \ldots, r$, then sum the results over all k to get $\mu(n)$. Write the resulting sum in your formula for $\mu(n)$ in a nice, compact, closed form and simplify.

8. *(Extra credit)* A strictly increasing sequence $\{f(n)\}$ of positive integers satisfies the recurrence relation

$$f(f(n)) = 3n.$$

Find $f(304)$. *Hint:* Don't use generating functions. Start by finding $f(f(1))$ and $f(1)$, then use the recurrence relation and the fact that each term must be greater than the previous one to compute first few terms of the sequence (up to $f(6)$ at least, since the next key idea after $f(1)$ is $f(4)$ and $f(5)$). By the time you compute all terms up to $f(9)$ (actually, up to $f(18)$ may be even better), you should have the idea as to how the whole sequence behaves.