Math 301A **Test 2** **04/12/02** **Name**______________________________

This exam is **due Monday, April 15, in class**. You may consult the text for this course (or *Topics in Algebra* by I.N. Herstein), a book on reserve for this course, your notes taken in lecture, your homework, and sketches of solutions of homework problems. Do not use other books or papers or materials from a library or consult with any person other than myself. There are some questions whose answers require rigorous argument. Show your argument *neatly* in the space provided. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the rules above. Remember to SHOW ALL WORK unless otherwise indicated.

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1. This problem consists of multiple-choice questions. Please *circle* the answer which is *always true* or *always correct*. Circle the *entire* answer, not just the letter labelling it. No work is required and no partial credit will be given for any part of this problem.

   (a) If G is a group of order 8740 and if H is a subgroup of G, then the following numbers are all possible as the number of cosets of H in G.

   i. $\{76, 92, 95, 391\}$
   ii. $\{40, 76, 115, 475\}$
   iii. $\{5, 95, 101, 437\}$
   iv. $\{4, 76, 95, 1748\}$
   v. $\{4, 95, 115, 2116\}$

   (b) Consider the group S_{35} of permutations of the set $\{1, 2, \ldots, 35\}$; so S_{35} is the symmetric group on 35 letters. Write H for the subgroup of all permutations $\sigma \in S_{35}$ such that $\sigma(9) = 9$ and $\sigma(17) = 17$. Then

   i. $(S_{35} : H) = 1190$
   ii. $(S_{35} : H) = 2$
   iii. $(S_{35} : H) = 33$
   iv. $(S_{35} : H) = 33!$
   v. $(S_{35} : H) = 1155$

   (c) Let G be a non-abelian finite group and let $\mathbb{Z}$ be the additive group of integers. Let $\phi : G \to \mathbb{Z}$ and $\psi : \mathbb{Z} \to G$ be two homomorphisms. Which of the following consists of *all* true statements?

   i. ϕ is always zero and ψ is never surjective.
   ii. ϕ is never injective and ψ is sometimes surjective.
   iii. ϕ is sometimes injective and ψ is sometimes surjective.
   iv. ϕ is sometimes surjective and ψ is sometimes injective.
   v. ϕ is never injective and ψ is always zero.

2. If G is a group, and if $\sigma, \tau \in G$, we define $[\sigma, \tau] = \sigma\tau\sigma^{-1}\tau^{-1} = (\sigma\tau)(\tau\sigma)^{-1} \in G$ (we call $[\sigma, \tau]$ the *commutator* of σ and τ). Let $\Delta(G)$ denote the subgroup of G generated by all these elements $[\sigma, \tau]$ as σ and τ vary over G. Of course, $[\sigma, \tau] \in \Delta(G)$ for all $\sigma, \tau \in G$.

   (a) Prove that $\Delta(G) \lhd G$. *Hint:* First prove $h[\sigma, \tau]h^{-1} \in \Delta(G)$ for any $h, \sigma, \tau \in G$.

   (b) Assuming $\Delta(G) \lhd G$, write $\overline{G} = G/\Delta(G)$. Prove that $\overline{G}$ is always an abelian group.

3. You are given G, a group of order 15, and it turns out that one can prove such a G must be abelian. (You may assume this.)

   (a) Without quoting Cauchy's theorem, prove directly that G possesses at least one element of order 3 and at least one element of order 5.

   (b) Assuming part (a), prove further that G is a cyclic group of order 15. *Hint:* Use elements from (a) to find an element of G of order 15.

4. Let $\phi : G \to G_1$ and $\psi : G \to G_2$ be two different homomorphisms. Suppose the *only* element of $x \in G$ for which $\phi(x) = 1_{G_1}$ and $\psi(x) = 1_{G_2}$ is $x = 1_G$. Prove: If $y \in \ker \phi$ and $z \in \ker \psi$, then $yz = zy$.

5. Let $\mathbb{F}_2$ be the set $\{0, 1\}$ with addition and multiplication defined as usual except for $1 + 1 = 0$ (i.e. $-1 = 1$). (Think of 0 as "even" and 1 as "odd" and of the facts about adding and multiplying even and odd numbers.) Let $GL(2, \mathbb{F}_2)$ be the group of 2×2 invertible matrices with coefficients in $\mathbb{F}_2$. Prove that $GL(2, \mathbb{F}_2) \simeq S_3$.