Math 301A **Sketch of Solutions for Test 1** **03/04/02**

1. Indicate whether each of the following statements is true or false. No work is required in this problem.

   (a) Since each permutation has an inverse, the product of all permutations in S_n is the identity permutation.

   FALSE; even S_2 disproves this.

   (b) If a permutation σ is of order 2, then σ is a product of disjoint transpositions.

   TRUE; if $\sigma^2 = e$ and $\sigma(i) = j$, then $\sigma(j) = i$.

   (c) Upper-triangular matrices with all 1's on the diagonal form a subgroup of $SL(n, \mathbb{R})$.

   TRUE; a product of upper-triangular matrices is upper triangular, products of diagonal entries are diagonal entries of the product.

   (d) $(\mathbb{Z}, *)$, where $a * b = ab + a + b$, is a group.

   FALSE; if true, then $0 * a = a * 0 = a$ means 0 is the identity element, but $(-1) * a = a * (-1) = -1 \neq 0$ for any integer a, so -1 does not have an inverse under $*$.

   (e) Let G be a group and H a subgroup of G. Define the centralizer $C(H)$ of H by $C(H) = \{g \in G \mid g^{-1}Hg = H\}$. Then $C(H) = \bigcap_{h \in H} C(h)$, where $C(h)$ is the centralizer of the element h.

   FALSE; $\bigcap_{h \in H} C(h)$ consists of elements $g \in G$ which fix each element of H (i.e. $g^{-1}hg = h$ for any $h \in H$). For g to be an element of $C(H)$, we only need to know $g^{-1}hg \in H$ for any $h \in H$, not necessarily $g^{-1}hg = h$ for any $h \in H$.

   (f) Not all groups of order 4 are cyclic.

   TRUE; e.g. the Klein's 4-group $V = \{e, x, y, z \mid x^2 = y^2 = z^2 = e,\ xy = yx = z,\ xz = zx = y,\ yz = zy = x\}$ is not cyclic.

   (g) A product of a 2-cycle and a 3-cycle is sometimes a 5-cycle.

   FALSE; either the cycles are disjoint, in which case their product is not a cycle, or they are not disjoint, in which case they move at most $2 + 3 - 1 = 4$ elements, hence the product cannot be a 5-cycle.

2. Let G be a group generated by two elements σ and τ. (In other words, G is the smallest group that contains σ and τ.)

   (a) If α is an element of G, explain carefully how you can conclude that α lies in the center $Z(G)$ of G provided you know that $\alpha\sigma = \sigma\alpha$ and $\alpha\tau = \tau\alpha$. (*Hint:* Use induction.)

   We need to prove that, for any $g \in G$, if $\alpha\sigma = \sigma\alpha$ and $\alpha\tau = \tau\alpha$, then $\alpha g = g\alpha$. Each element $g \in G$ is a product of a string of $n \geq 0$ factors each of which is σ, τ, σ^{-1} or τ^{-1}. Our proof will be by induction on n. The implication is clearly true for $g = e$, $g = \sigma$ and $g = \tau$. Also, $\alpha\sigma = \sigma\alpha$ implies $\sigma^{-1}\alpha\sigma\sigma^{-1} = \sigma^{-1}\sigma\alpha\sigma^{-1}$, i.e. $\sigma^{-1}\alpha = \alpha\sigma^{-1}$ Similarly, $\alpha\tau = \tau\alpha$ implies $\tau^{-1}\alpha\tau\tau^{-1} = \tau^{-1}\tau\alpha\tau^{-1}$, i.e. $\tau^{-1}\alpha = \alpha\tau^{-1}$. Thus, our statement is true for $g = \sigma^{-1}$ and $g = \tau^{-1}$ as well, in other words, if $n = 0, 1$. Suppose now that our statement is true for products g' of at most $n - 1$ factors from $\{\sigma, \tau, \sigma^{-1}, \tau^{-1}\}$. Let g be a product of n factors from $\{\sigma, \tau, \sigma^{-1}, \tau^{-1}\}$. Then $g = g'x$, where g' is as above and $x \in \{\sigma, \tau, \sigma^{-1}, \tau^{-1}\}$, so we have $g'\alpha = \alpha g'$ by induction hypothesis, and hence, $g\alpha = g'x\alpha = g'\alpha x = \alpha g'x = \alpha g$. Thus, our statement is true for any n, i.e. for any element $g \in G$.

   (b) Now suppose that G is generated by σ and τ, and that $\sigma^5 = 1$, $\tau^3 = 1$, and $\sigma^4\tau = \tau\sigma$. Show that τ^2 lies in the center of G. (*Hint:* Use part (a).)

   By part (a), we only need to show that τ^2 commutes with the generators σ and τ. Obviously, $\tau^2\tau = \tau^3 = \tau\tau^2$, so τ^2 commutes with τ. Also, $\sigma^4 = \sigma^5\sigma^{-1} = 1\sigma^{-1} = \sigma^{-1}$, so $\tau\sigma = \sigma^{-1}\tau$, hence $\sigma\tau\sigma = \tau$, so $\sigma\tau = \tau\sigma^{-1}$. Therefore, $\tau^2\sigma = \tau\tau\sigma = \tau\sigma^{-1}\tau = \sigma\tau\tau = \sigma\tau^2$, so τ^2 commutes with σ as well. Thus, by part (a) $\tau^2 \in Z(G)$.

(c) Now show that τ itself lies in the center of G.

Recall that the center $Z(G)$ of G is a subgroup of G. From part (b), $\tau^2 \in Z(G)$, so $\tau = 1\tau = \tau^3\tau = \tau^4 = \tau^2\tau^2 \in Z(G)$.

(d) From your results above and $\sigma^4\tau = \tau\sigma$, conclude that $\sigma^3 = 1$.

From part (c), $\tau \in Z(G)$, so τ commutes with any element of G, in particular, $\tau\sigma = \sigma\tau$. Thus, $\sigma^3\sigma\tau = \sigma^4\tau = \tau\sigma = \sigma\tau$, so $\sigma^3 = 1$.

(e) Finally, show that G is actually generated by τ alone, and G is in fact cyclic of order 3.

From part (d), $\sigma^3 = 1$, so $1 = (\sigma^3)^2 = \sigma^6 = \sigma^5\sigma = 1\sigma = \sigma$. Thus, $\sigma = 1$. Since $\sigma = 1$, and any $g \in G$ is a product of some string of σ's or τ's, we obtain that each $g \in G$ is a product of a string of τ's, i.e. G is generated by τ alone, i.e. $G = (\tau)$ and hence G is cyclic. But $\tau^3 = 1$, so the distinct elements of G are $1, \tau, \tau^2$. Hence G is cyclic of order 3.

3. Show that S_n is generated by transpositions of consecutive integers, i.e. (12), (23), (34), $\ldots$, $(n-1\ n)$. *Hint:* First show that any transposition (ij), where $1 \le i < j \le n$, can be represented as a product of the transpositions above, e.g. $(13) = (12)(23)(12)$.

In fact, the above example can be generalized: for any three distinct integers i, j, k, we have $(ik) = (ij)(jk)(ij)$. In particular, $(i\ k) = (i\ i+1)(i+1\ k)(i\ i+1)$ for $k > i+1$. A simple proof by induction on $k - i$ (a rigorous way of applying the above formula $k - i$ times to the middle element) shows that this implies

$$(i\ k) = (i\ i+1)(i+1\ i+2)\ldots(k-2\ k-1)(k-1\ k)(k-2\ k-1)\ldots(i+1\ i+2)(i\ i+1).$$

Therefore, any transposition is a product of transpositions of consecutive elements. Since each permutation is a product of transpositions, each permutation is a product of transpositions of consecutive elements. In other words, G is generated by $(12), (23), (34), \ldots, (n-1\ n)$.

4. (a) Consider the group $D_4 = \{e, \psi, \psi^2, \psi^3, \phi, \phi\psi, \phi\psi^2, \phi\psi^3 \mid \phi^2 = \psi^4 = e, \psi\phi = \phi\psi^{-1}\}$. What is the centralizer of ϕ?

Since there are only 8 elements, one can simply check which ones commute with ϕ. Those turn out to be $e, \phi, \psi^2, \phi\psi^2$. Thus, $C(\phi) = \{e, \phi, \psi^2, \phi\psi^2\}$.

(b) Consider the group $D_5 = \{e, \psi, \psi^2, \psi^3, \psi^4, \phi, \phi\psi, \phi\psi^2, \phi\psi^3, \phi\psi^4 \mid \phi^2 = \psi^5 = e, \psi\phi = \phi\psi^{-1}\}$. What is the centralizer of ϕ?

Similarly, we can check that only e and ϕ commute with ϕ. Thus, $C(\phi) = \{e, \phi\}$.

(c) (extra) Find the centralizer of ϕ in $D_n = \langle \phi, \psi \mid \phi^2 = \psi^n = e, \psi\phi = \phi\psi^{-1} \rangle$. *Hint:* The answer depends on whether n is even or odd.

Of course, checking each integer n is not very efficient, so we need another solution. Obviously, any power of ϕ commutes with ϕ, hence $e, \phi \in C(\phi)$. Does any power ψ^k $(0 \le k \le n-1)$ of ψ other than e commute with ϕ? If it does, then $\phi\psi^k = \psi^k\phi$. But $\psi\phi = \phi\psi^{-1}$, so

$$\psi^k\phi = \underbrace{\psi\ldots\psi}_{k}\phi = \underbrace{\psi\ldots\psi}_{k-1}\phi\psi^{-1} = \cdots = \phi\underbrace{\psi^{-1}\ldots\psi^{-1}}_{k} = \phi\psi^{-k}.$$

Hence, $\phi\psi^k = \psi^k\phi = \phi\psi^{-k}$, so $\psi^k = \psi^{-k}$, i.e. $\psi^{2k} = e$, so $2k$ is divisible by n. Since $0 \le 2k \le 2n-2$, we have $2k = 0$ (i.e. $k = 0$) or $2k = n$. The solution $k = 0$ yields $\psi^0 = e$, so we need $k = n/2$, which is an integer if and only if n is even. Thus, if n is odd, only e and ϕ commute with ϕ, so $C(\phi) = \{e, \phi\}$ for odd n. If n is even, then $e, \phi, \psi^{n/2} \in C(\phi)$, so $\phi\psi^{n/2} \in C(\phi)$ as well, since $C(\phi)$ is a group. Solving $\phi(\phi\psi^l) = (\phi\psi^l)\phi$, we similarly get $l = n/2$, so we found all the elements which commute with ϕ. Thus, $C(\phi) = \{e, \phi, \psi^{n/2}, \phi\psi^{n/2}\}$ for even n.