Math 301A **Test 1** **03/01/02** **Name**____________________________

This exam is **due Monday, March 4, in class**. You may consult the text for this course (or *Topics in Algebra* by I.N. Herstein), a book on reserve for this course, your notes taken in lecture, your homework, and sketches of solutions of homework problems. Do not use other books or papers or materials from a library or consult with any person other than myself. There are some questions whose answers require rigorous argument. Show your argument *neatly* in the space provided. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the rules above. Remember to SHOW ALL WORK unless otherwise indicated.

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1. Indicate whether each of the following statements is true or false. No work is required in this problem.

   (a) Since each permutation has an inverse, the product of all permutations in S_n is the identity permutation.

   (b) If a permutation σ is of order 2, then σ is a product of disjoint transpositions.

   (c) Upper-triangular matrices with all 1's on the diagonal form a subgroup of $SL(n, \mathbb{R})$.

   (d) $(\mathbb{Z}, *)$, where $a * b = ab + a + b$, is a group.

   (e) Let G be a group and H a subgroup of G. Define the centralizer $C(H)$ of H by $C(H) = \{g \in G \mid g^{-1}Hg = H\}$. Then $C(H) = \bigcap_{h \in H} C(h)$, where $C(h)$ is the centralizer of the element h.

   (f) Not all groups of order 4 are cyclic.

   (g) A product of a 2-cycle and a 3-cycle is sometimes a 5-cycle.

2. Let G be a group generated by two elements σ and τ. (In other words, G is the smallest group that contains σ and τ.)

   (a) If α is an element of G, explain carefully how you can conclude that α lies in the center $Z(G)$ of G provided you know that $\alpha\sigma = \sigma\alpha$ and $\alpha\tau = \tau\alpha$. (*Hint:* Use induction.)

   (b) Now suppose that G is generated by σ and τ, and that $\sigma^5 = 1$, $\tau^3 = 1$, and $\sigma^4\tau = \tau\sigma$. Show that τ^2 lies in the center of G. (*Hint:* Use part (a).)

   (c) Now show that τ itself lies in the center of G.

(d) From your results above and $\sigma^4\tau = \tau\sigma$, conclude that $\sigma^3 = 1$.

(e) Finally, show that G is actually generated by τ alone, and G is in fact cyclic of order 3.

3. Show that S_n is generated by transpositions of consecutive integers, i.e. (12), (23), (34), ..., $(n-1\ n)$. *Hint:* First show that any transposition (ij), where $1 \leq i < j \leq n$, can be represented as a product of the transpositions above, e.g. $(13) = (12)(23)(12)$.

4. (a) Consider the group $D_4 = \{e, \psi, \psi^2, \psi^3, \phi, \phi\psi, \phi\psi^2, \phi\psi^3 \mid \phi^2 = \psi^4 = e, \psi\phi = \phi\psi^{-1}\}$. What is the centralizer of ϕ?

(b) Consider the group $D_5 = \{e, \psi, \psi^2, \psi^3, \psi^4, \phi, \phi\psi, \phi\psi^2, \phi\psi^3, \phi\psi^4 \mid \phi^2 = \psi^5 = e, \psi\phi = \phi\psi^{-1}\}$. What is the centralizer of ϕ?

(c) (extra) Find the centralizer of ϕ in $D_n = \langle \phi, \psi \mid \phi^2 = \psi^n = e, \psi\phi = \phi\psi^{-1} \rangle$. *Hint:* The answer depends on whether n is even or odd.