Part A.

1. (a) *Closure:* If $(h,k),(h',k') \in H\times K$, then $h,h' \in H$ and $k,k' \in K$, so $hh' \in H$ and $kk' \in K$, so $(h,k)(h',k') = (hh',kk') \in H \times K$.
Associativity: $((h,k)(h',k'))(h",k") = (hh',kk')(h",k") = (hh'h",kk'k") =$
$= (h,k)(h'h",k'k") = (h,k)((h',k')(h",k"))$.
Identity: $(h,k)(e,e) = (he,ke) = (h,k) = (eh,ek) = (e,e)(h,k)$, so $(e,e) = 1_{H\times K}$.
Inverse: $(h,k)(h^{-1},k^{-1}) = (hh^{-1},kk^{-1}) = (e,e) = (h^{-1}h,k^{-1}k) = (h^{-1},k^{-1})(h,k)$, so $(h,k)^{-1} = (h^{-1},k^{-1}) \in H \times K$.

   (b) Since $hk = kh$ for any $h \in H$ and $k \in K$, $HK = KH$, hence HK is a group. Consider a map $f : H \times K \to HK$ given by $f(h,k) = hk$. Then $f((h,k)(h',k')) = f(hh',kk') = hh'kk' = hkh'k' = f(h,k)f(h'k')$, so f is a homomorphism. Any element in HK can be written as $hk = f(h,k)$ for some $h \in H$ and $k \in K$, so f is onto. By the First Homomorphism Theorem, $HK = (H\times K)/\ker(f)$. What is $\ker(f)$? If $h \in H$ and $k \in K$, then $(h,k) \in \ker(f)$ if and only if $hk = f(h,k) = e$, i.e. if and only if $k = h^{-1}$. But then $k \in H$ and $h \in K$, so $h,k \in H \cap K$. Thus, $\ker(f) = \{(h,h^{-1}) \mid h \in H \cap K\}$ which is obviously isomorphic to $H \cap K$ via the isomorphism $h \mapsto (h,h^{-1})$ (check that this is an isomorphism).

   (c) Consider the same map $f : H \times K \to HK$, $f(h,k) = hk$. How many elements are in the inverse image of a given element $hk \in HK$? We have $f(h',k') = hk \iff h'k' = hk \iff h^{-1}h' = k(k')^{-1}$. Let $a = h^{-1}h' = k(k')^{-1}$, then $a \in H$ and $a \in K$, so $a \in H \cap K$. Then $h' = ha$ and $k' = a^{-1}k$, so $(h',k') = (ha,a^{-1}k)$ for some $a \in H \cap K$. Conversely, for any $a \in H \times K$, $f(ha,a^{-1}k) = haa^{-1}k = hk = f(h,k)$. Thus, the inverse image of hk is $f^{-1}(hk) = \{(ha,a^{-1}k) \mid a \in H \cap K\}$. Thus, there is a bijection between $f^{-1}(hk)$ and $H \cap K$ given by $a \mapsto (ha,a^{-1}k)$, so $f^{-1}(hk)$ has $o(H \cap K)$ elements. Since any element of HK can be written as hk for some $h \in H$ and $k \in K$, the inverse image of any element of HK under f has $o(H\cap K)$ elements. In other words, $o(H)o(K) = o(H \times K) = o(HK)o(H \cap K)$, so $o(HK) = o(H)o(K)/o(H \cap K)$.
Note: This holds even if HK is not a group.

   (d) Suppose $H \lhd G$ and $K \lhd G$. Then for any $g \in G$, we have $gHg^{-1} = H$ and $gKg^{-1} = K$, so $gHKg^{-1} = gHg^{-1}gKg^{-1} = HK$, i.e. $HK \lhd G$.

2. (a) Suppose that $a,b \in N(H)$, then $a^{-1}Ha = H$ and $b^{-1}Hb = H$, so $(ab)^{-1}Hab = b^{-1}a^{-1}Hab = b^{-1}Hb = H$, hence $ab \in N(H)$. If $a \in N(H)$, then $a^{-1}Ha = H$, so $H = aHa^{-1} = (a^{-1})^{-1}Ha^{-1}$, so $a^{-1} \in N(H)$. Thus, $N(H)$ is a subgroup of G.

   (b) For any element $a \in N(H)$, we have $a^{-1}Ha = H$, hence $H \lhd N(H)$.

   (c) If K is a subgroup of G such that $H \lhd K$, then for any $k \in K$, $k^{-1}Hk = H$, so $k \in N(H)$. Therefore, $K \subseteq N(H)$.

Part B.

1. We know that $A' \lhd A$ and $B' \lhd B$, so by the Second Homomorphism Theorem, we have that $A' \cap C \lhd A \cap C = C$ and $C \cap B' \lhd C \cap B = C$. Hence, by Problem A2(d), $(A' \cap C)(C \cap B') \lhd C$. Let $C' = (A' \cap C)(C \cap B')$.

   Since $A' \lhd A$, it follows that $A'C$ is a subgroup of $AC = A(A \cap B) = A \cap AB = A$. Consider a map from $f : A'C \to C/C'$ given by $f(a'c) = cC'$ for any $a' \in A'$ and $c \in C$. Then f

is well-defined, since $a'c = a_1'c_1 \iff a'(a_1')^{-1} = c^{-1}c_1 \in A' \cap C \subseteq (A' \cap C)(C \cap B') = C' \iff cC' = c_1C'$. Also, for any $c \in C$, $cC' = f(ec)$, and $ec \in A'C$, so f is onto. Finally, if $a'c, a_1'c_1 \in A'C$, then $(a'c)(a_1'c_1) = a'(ca_1'c^{-1})cc_1$, and $ca_1'c^{-1} \in A'$ since $A' \lhd A$ and $c \in C \subseteq A$. Thus, $f((a'c)((a_1'c_1)) = f(a'(ca_1'c^{-1})cc_1) = cc_1C' = (cC')(c_1C') = f(a'c)f(a_1'c_1)$, so f is a homomorphism. Hence, $C/C' \simeq (A'C)/\ker(f)$ by the First Homomorphism Theorem. What is $\ker(f)$? We have $a'c \in \ker(f) \iff f(a'c) = C' \iff cC' = C' \iff c \in C'$. But then $a'c \in A'(A' \cap C)(C \cap B')$, and $A'(A' \cap C) = A' \cap A'C = A'$, so $a'c \in A'(C \cap B')$. Therefore, $\ker(f) \subseteq A'(C \cap B')$. On the other hand, any element in $A'(C \cap B')$ is of the form $a'c$ for some $a' \in A'$ and $c \in C \cap B' \subseteq (A' \cap C)(C \cap B') = C'$, so $f(a'c) = cC' = C'$, i.e. $a'c \in \ker(f)$. Therefore, $A'(C \cap B') \subseteq \ker(f)$. Thus, $\ker(f) = A'(C \cap B')$, so $C/C' \simeq (A'C)/A'(C \cap B')$ and $A'(C \cap B') \lhd A'C$. Switching A and B as well as A' and B', we get $C/C' \simeq (B'C)/B'(A' \cap C)$ and $B'(A' \cap C) \lhd B'C$.

2. For $\sigma \in G$, we have $\sigma = \begin{pmatrix} 1/c & b \\ 0 & c \end{pmatrix}$ for some $c > 0$ and some real b. Let $(x, y) \in \mathbb{R}^2$. From the definition of the action of G on $\mathbb{R}^2$, it is clear that the action of any $\sigma \in G$ preserves the sign of y, and if $y = 0$ then the action of any $\sigma \in G$ preserves the sign of x.

   If $y > 0$, then
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/y & x \\ 0 & y \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \qquad \begin{pmatrix} 1/c & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix}, \quad c > 0.$$
   If $y < 0$, then
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/y & -x \\ 0 & -y \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix}; \qquad \begin{pmatrix} 1/c & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -b \\ -c \end{pmatrix}, \quad -c < 0.$$
   If $y = 0$ and $x > 0$, then
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad a > 0.$$
   If $y = 0$ and $x < 0$, then
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} -x & 0 \\ 0 & -1/x \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix}; \qquad \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -a \\ 0 \end{pmatrix}, \quad -a < 0.$$
   If $y = 0$ and $x = 0$, then
$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$
   Thus, the upper half-plane, $y > 0$, is the orbit of $(0, 1)$; the lower half-plane, $y < 0$, is the orbit of $(0, -1)$; the positive x-axis, $y = 0$ and $x > 0$, is the orbit of $(1, 0)$; the negative x-axis, $y = 0$ and $x < 0$, is the orbit of $(-1, 0)$; and the origin, $(0, 0)$, is in a separate orbit by itself. Altogether, there are 5 orbits.

3. Consider the action of G on the set of its subgroups by conjugation. In other words, $\sigma * H = \sigma H \sigma^{-1}$ for $\sigma \in G$ and a subgroup H of G. Then the orbit of any subgroup H of G is the set of all subgroups conjugate to H, and the stabilizer of H is the set of elements $\sigma \in G$ such that $\sigma H \sigma^{-1} = H$, i.e. $N(H)$. By the orbit-stabilizer theorem, the number of elements in the orbit of H is the index of its stabilizer in G. In our case this means that the number of subgroups conjugate to H is $i_G(N(H))$.