Math 301 **Assignment 4** **Due 04/29/02**

For fun: Do problems from Chapter 2, sections 8, 9, 10, 11. These don't have to be turned in, but solving them will give you valuable experience and insight you will need later on in the course. Feel free to ask me questions about these problems during my office hours.

Part A.

1. Let G be a group and let $H, K \subseteq G$ be two subgroups of G. The *direct product* $H \times K$ of H and K is defined as the set of ordered pairs (h, k), where $h \in H$ and $k \in K$, i.e. $H \times K = \{(h, k) \mid h \in H,\ k \in K\}$, together with the component-wise product $(h, k)(h', k') = (hh', kk')$ for any $h \in H$ and $k \in K$.

   (a) Show that $H \times K$ is a group.

   (b) Suppose $hk = kh$ for any $h \in H$ and $k \in K$. Show that
   $$HK \simeq (H \times K)/(H \cap K).$$
   Hint: Find a homomorphism from one of these groups to another with an appropriate kernel.

   (c) Prove that
   $$o(KH) = o(HK) = \frac{o(H)o(K)}{o(H \cap K)}$$
   for any finite subgroups H and K of G (so HK and KH do not have to be groups, just subsets of G).
   Hint: You don't even need a homomorphism here, just a map from one set onto another so that each element in the target has the inverse image of the same size.

   (d) Suppose $H \lhd G$ and $K \lhd G$. Show that $HK \lhd G$.

2. If G is a group and H is a subgroup of G, let $N(H) = \{a \in G \mid a^{-1}Ha = H\}$. $N(H)$ is called the *normalizer*, or the *centralizer*, of H. (Recall that $N(H)$ was also denoted $C(H)$ in class.)

   (a) Show that $N(H)$ is a subgroup of G.

   (b) Show that $H \lhd N(H)$.

   (c) Show that if K is a subgroup of G such that $H \lhd K$, then $K \subseteq N(H)$ (so $N(H)$ is the largest subgroup of G in which H is normal).

Part B.

1. Let A and B be subgroups of G, $C = A \cap B$, and let $A' \lhd A$, $B' \lhd B$. Show that $A'(C \cap B') \lhd A'C$, $B'(A' \cap C) \lhd B'C$, $(C \cap B')(A' \cap C) \lhd C$, and
   $$[A'C]/[A'(C \cap B')] \simeq C/[(C \cap B')(A' \cap C)] \simeq [B'C]/[B'(A' \cap C)].$$

Remark: The claim in the above problem is called Zassenhaus' Lemma or the Butterfly Lemma. If $B \subseteq A$ and $B' = (e)$, we get the Second Homomorphism Theorem. Also note that $C \cap B' = A \cap B'$ and $A' \cap C = A' \cap B$.

Hint: Use the Second Homomorphism Theorem and Problem A1 to prove that $(C \cap B')(A' \cap C) \lhd C$. Then prove the two isomorphisms in a way analogous to the proof of the Third Homomorphism Theorem.

2. Consider the group G consisting of all 2×2 upper triangular real matrices $\sigma = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ for which $\det(\sigma) = 1$ and $a > 0$. Let G act on the the xy-plane (i.e. on $S = \mathbb{R}^2$) according to the rule:

$$\sigma * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cy \end{pmatrix}$$

   Into how many orbits does the action of G partition the xy plane? What are they?

3. Show that the number of subgroups of G conjugate to H is $i_G(N(H))$, the index of $N(H)$ in G. (*Hint:* Consider the action of G on the set of its subgroups by conjugation and use the orbit-stabilizer theorem.)

Part C. (extra)

1. Let G be a finite group, and write $c(G)$ for the number of conjugacy classes of G. In general, this number will increase as $o(G) \to \infty$, so we introduce the number

$$\gamma(G) = \frac{c(G)}{o(G)}.$$

   Clearly, $0 < \gamma(G) \leq 1$, and $\gamma(G) = 1$ if and only if G is abelian. Assume from now on that G is non-abelian.

   (a) Prove that $\gamma(G) \leq 5/8$.

   (b) If p is the smallest prime which divides $o(G)$, prove that

$$\gamma(G) \leq \frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^3}.$$

   (c) Are the above bounds sharp? That is, can you find a group G with $\gamma(G) = 5/8$? similarly, for (b)?

   Hint: Use the Class Equation. Consider the center of G and the rest of G.

2. Let G be a group of order pqr where $p < q < r$ are prime numbers.

   (a) Show that either the q-Sylow subgroup of G or the r-Sylow subgroup of G is normal in G.

   (b) Show that the r-Sylow subgroup of G is normal in G.