Math 301 **Assignment 3** **Due 04/05/02**

For fun: Do problems from Chapter 1, section 5, and Chapter 2, sections 4, 5, 6, 7. These don't have to be turned in, but solving them will give you valuable experience and insight you will need later on in the course. Feel free to ask me questions about these problems during my office hours.

In some problems below, equalities and inequalities between numbers are asked for. In these problems, some of the numbers may well be infinite; so, we make the following *convention*:

1. That an inequality $x \leq y$ holds means *three* things:
   (a) if x and y are finite, the usual meaning that $x \leq y$;
   (b) if y is finite, the assertion is that x is finite and (1a) holds;
   (c) if x is infinite, the assertion is that y must be infinite.
   Note: (1b) $\implies$ (1c).
2. That an equality $x = y$ holds means two things:
   (a) if x and y are finite, the usual meaning that $x = y$;
   (b) if y is finite then so is x; if x is finite then so is y; then (2a) also holds;

Part A.

1. Let G be any group and let H and K be subgroups of G. Assume $K \subseteq H$. Prove the generalized Lagrange's Theorem:
$$i_G(K) = i_G(H)i_H(K),$$
   where $i_G(H)$, or $(G : H)$, the *index* of H in G, is the number of right cosets of H in G.
2. Let H be a subgroup of a group G, and write X for the set of right cosets, G/H, of H in G. If $\tau \in G$, and $x \in X$ is a right coset, pick some $\sigma \in x$. Then $x = H\sigma$. Define $x * \tau$ to be the right coset of $\sigma\tau$, i.e. in symbols: $(H\sigma) * \tau = H(\sigma\tau)$.
   (a) Show that this is well defined (i.e. does not depend on the choice of σ).
   (b) For $x \in X$, let $St(x) = \{\tau \in G \mid x * \tau = x\}$. We call $St(x)$ the *stabilizer* of x. Show that $St(x) = \sigma^{-1}H\sigma$. Furthermore, show that $N = \bigcap_{\sigma \in G} \sigma^{-1}H\sigma$ acts trivially on X, i.e. $x * \tau = x$ for any $\tau \in N$ and any $x \in X$.
   (c) Write $A(X)$ for the group of 1-1 maps of X onto itself. The element τ, via the action $*$, gives an element $*(\tau) = a_\tau \in A(X)$. Namely,
$$a_\tau(H\sigma) = H(\sigma\tau).$$
   Prove that $* : \tau \mapsto a_\tau$ is a homomorphism from G to $A(X)$. What is $\ker(*)$?
   (d) $i_G(H) = n$, then $A(X)$ has $n!$ elements. Prove that if $o(G) > n!$ and G has a subgroup of index n, then G has a nontrivial normal subgroup, i.e. G is not simple.
   Remark: This means a simple group can't have subgroups "too close to the top".

Part B.

1. Let G be any group and let H and K be subgroups of G. Prove that

$$i_G(H \cap K) \leq i_G(H) i_G(K).$$

Hence, (Poincaré) the intersection of subgroups of finite index in G is itself of finite index.

2. Write $\mathbb{N}$ for the set of natural (=counting) numbers, $\mathbb{N} = \{1, 2, 3, 4, \dots\}$. Let

$$S_\infty = \{\sigma \in A(\mathbb{N}) \mid \exists M \text{ (perhaps depending on } \sigma) \text{ so that } \sigma(n) = n \text{ for all } n \geq M\}$$

In other words, S_∞ is the set of all those permutations of $\mathbb{N}$ that leave all but finitely many integers fixed.

   (a) Prove that S_∞ is a subgroup of $A(\mathbb{N})$.

   (b) Show that the notion of even permutation makes sense (is well-defined) in S_∞. Call the set of even permutations, A_∞, the *infinite alternating group* (it *is* a subgroup of S_∞). Prove that $A_\infty \lhd S_\infty$ and $(S_\infty : A_\infty) = 2$.

   (c) It is a famous theorem of E. Galois that A_n is simple (has no nontrivial normal subgroups) for $n \geq 5$. Assume this and prove that A_∞ is also simple.

   (d) Show that S_∞ possesses exactly two subgroups of finite index (trivial or nontrivial). However, exhibit an infinite number of finite subgroups of S_∞. Indeed, if p is a prime number and $r > 0$ is an integer, show that S_∞ has an infinite number of distinct subgroups of exact order p^r.

   (e) Explain why every finite group is isomorphic to a subgroup of S_∞. Plainly, S_∞ is a very complicated infinite group.

   (f) Even though S_∞ is big and complicated, prove that neither $\mathbb{Z}$ nor $SL(2, \mathbb{Z})$ are isomorphic to subgroups of S_∞.

   (g) Lastly, prove: If H is a subgroup of S_∞ and H is finitely generated (has finitely many generators), then H is a finite group.

3. (a) Let G be a *finite* group. Suppose there exists a automorphism ϕ of G having no fixed points except 1, i.e. $\phi(x) = x \implies x = 1$. If $\phi^2 = id_G$, prove that G must be abelian.

   Hint: Begin by proving $(\forall x \in G)(\exists y \in G)(yx = \phi(y))$. You won't need $\phi^2 = id_G$ for this.

   (b) The result above is *false* if G is not finite. Find an infinite group G which is *non-abelian*, having an automorphism ϕ as above.