Math 301 **Assignment 2** **Due 02/22/02**

For fun: Do problems from Chapter 2, sections 1, 2, 3. These don't have to be turned in, but solving them will give you valuable experience and insight you will need later on in the course. Feel free to ask me questions about these problems during my office hours.

Part A.

1. (a) Let G be a group. Suppose that $x^2 = 1_G$ for any $x \in G$. Show that G is abelian.

   Remark: 1_G is another notation for the identity element of G. In fact, if no confusion may arise in a problem under consideration, we may drop the subscript and just write 1 for 1_G.

   (b) Let G be a group. Suppose that for some integer $n \geq 0$, we have

   $$(xy)^n = x^n y^n, \quad (xy)^{n+1} = x^{n+1} y^{n+1}, \quad (xy)^{n+2} = x^{n+2} y^{n+2}$$

   for any $x, y \in G$. Show that G is abelian.

   (c) Give an example of a nonabelian group G such that for some integer $n > 0$,

   $$(xy)^n = x^n y^n, \quad (xy)^{n+1} = x^{n+1} y^{n+1}$$

   for any $x, y \in G$.

2. Let G be a group, H be a subgroup of G.

   (a) For a given $x \in G$, let $x^{-1}Hx = \{x^{-1}hx \mid h \in H\}$. Show that $x^{-1}Hx$ is a subgroup of G for any $x \in G$.

   Remark: K is called a subgroup *conjugate* to H if $K = x^{-1}Hx$ for some $x \in G$.

   (b) Let

   $$N = \bigcap_{x \in G} x^{-1}Hx.$$

   Prove that N is a subgroup of G such that $y^{-1}Ny = N$ for any $y \in G$.

   Remark: A subgroup N of G is called *normal* if $y^{-1}Ny = N$ for any $y \in G$. If N is a normal subgroup of G, we write $N \triangleleft G$.

Part B.

1. Let G be a set with an operation $*$ such that:

   (a) G is closed under $*$.

   (b) $*$ is associative.

   (c) There is an element $e \in G$ such that $e * x = x$ for all $x \in G$. (e is called a left identity).

   (d) Given any $x \in G$, there exists an element $inv(x) \in G$ (which depends on x) such that $inv(x) * x = e$. ($inv(x)$ is called a left inverse of x.)

   Show that G is a group. (So you must prove that $x * inv(x) = e$ and $x * e = x$ for $e, x, inv(x)$ as above).

   Hint: Consider x, $inv(x)$, $inv(inv(x))$ and some cleverly constructed products of those. Do not assume $*$ is commutative! First show a left inverse is also a right inverse, then show a left identity is also a right identity.

2. Let $GL(2, \mathbb{C})$ be the group of 2×2 invertible complex matrices. Write

   $$\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad i = \sqrt{-1}.$$

   Let D be the smallest subgroup of $GL(2, \mathbb{C})$ containing α and β. Let H be the smallest subgroup of $GL(2, \mathbb{C})$ containing α and γ.

   (a) Prove that H has exactly 8 elements $\{\pm I, \pm x, \pm y, \pm z\}$ which satisfy

   $$x^2 = y^2 = z^2 = -I, \quad xy = z, \quad yz = x, \quad zx = y.$$

   (b) Is D finite? If so, how many elements are in D? How is D related to the group D_4 of symmetries of a square?

   (c) Show that there is no bijection $\varphi : D \to H$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for any $a, b \in D$.

   Hint: Find a property of D which any such φ preserves but H does not have. A function $f : (G_1, *) \to (G_2, \star)$ such that $f(a * b) = f(a) \star f(b)$ for any $a, b \in G_1$ is called a *homomorphism*. A homomorphism which is also a bijection is called an *isomorphism*.

3. Let $SL(2, \mathbb{Z})$ denote the group of 2×2 integer matrices whose determinant is 1. (Check that $SL(2, \mathbb{Z})$ is indeed a group, whereas $GL(2, \mathbb{Z})$ is not.)

   (a) Find matrices $x, y \in SL(2, \mathbb{Z})$ such that $x^4 = y^6 = I$ but $(xy)^n \neq I$ for any integer $n \neq 0$.

   (b) (extra) Choose your x, y as above so that they also generate $SL(2, \mathbb{Z})$, i.e. every $z \in SL(2, \mathbb{Z})$ can be expressed as a product of a string of x's and y's.

   Hint: Look for simplest examples. Search among matrices with entries equal to $0, 1, -1$. In part (b), make use of the Euclidean algorithm described in Chapter 1, section 5, pp. 21-26.