

**Part A.**

1. (a) The empty set,  $\emptyset$ , has no nonempty proper subset (since it has no nonempty subset), hence no bijection to a nonempty proper subset. Thus, the empty set is finite.
- (b) We will prove the contrapositive: "If  $S$  is a set and  $T \subseteq S$ , then  $S$  is infinite if  $T$  is infinite."

Indeed, suppose  $T$  is infinite. Then there is a nonempty subset  $T_0 \subseteq T$  such that  $T_0$  is in 1-1 correspondence with  $T$ . Let  $\varphi : T \rightarrow T_0$  be a bijection from  $T$  to  $T_0$ . Define a function  $f$  on  $S$  as follows:

$$f(s) = \begin{cases} \varphi(s) & \text{if } s \in T, \\ s & \text{if } s \in S - T. \end{cases}$$

In other words,  $f$  is  $\varphi$  on  $T$  and the identity function on  $S - T$ . Let  $S_0 = (S - T) \cup T_0$ . Then  $S_0 = f(S)$  and  $f$  is a bijection from  $S$  to  $S_0$ , since

$$f^{-1}(s) = \begin{cases} \varphi^{-1}(s) & \text{if } s \in T, \\ s & \text{if } s \in S - T. \end{cases}$$

is also a function. It is also easy to show that  $S - S_0 = T - T_0 \neq \emptyset$ . Thus, there is a bijection from  $S$  to its nonempty proper subset, so  $S$  is infinite.

2. Part (d) should have had "is divisible by  $n$ " in place of "is divides  $n$ ." The answers are as follows:

Part	Reflexive	Symmetric	Transitive	Equivalence
a	Y	Y	Y	Y
b	N	Y	N	N
c	Y	N	Y	N
d	Y	Y	Y	Y
e	Y	Y	Y	Y
f	N	Y	N	N
g	Y	Y	Y	Y

**Part B.**

1. Problems A1 and B2 will both be useful here, so read their solutions first. Our proof is by contradiction. Assume  $R$  and  $S$  are finite, but  $T = R \cup S$  is infinite. Then there is a bijection  $\varphi$  from  $T$  to some nonempty proper subset  $T_0$  of  $T$ .  $T_0 \subsetneq T$ , so  $T - T_0 \neq \emptyset$ . Let  $e \in T - T_0$ . Construct a subset

$E$  of  $T$  as in Problem B2, in other words,  $E = \{a_n \mid a_n = \varphi^{i-1}(e), i \in \mathbb{N}\}$ . Then  $E$  is infinite since  $\varphi(E) = E - \{e\}$ .

Consider two subsets of  $E$ :  $E_R = E \cap R$  and  $E_S = E \cap S$ . Then  $E_R \subseteq R$ ,  $E_S \subseteq S$ , and  $E_R \cup E_S = E$ . (Why?) Now consider the sets of indices  $\mathbb{N}_R = \{n \mid a_n \in E_R\}$  and  $\mathbb{N}_S = \{n \mid a_n \in E_S\}$  of elements of  $E$  that are in  $E_R$  and  $E_S$ , respectively. Then  $\mathbb{N}_R \cup \mathbb{N}_S = \mathbb{N}$ . Can both  $\mathbb{N}_R$  and  $\mathbb{N}_S$  have largest elements (say,  $N_1$  and  $N_2$ )? No, since  $N = \max(N_1, N_2)$  would have been the largest positive integer in that case. This is impossible since  $N + 1 > N$  is also a positive integer. Thus, one of the two sets (say,  $\mathbb{N}_R$ ) is nonempty and has no largest element. Let  $n_1$  be the smallest integer in  $\mathbb{N}_R$ ,  $n_2$  be the smallest integer in  $\mathbb{N}_R$  larger than  $n_1$ ,  $n_3$  be the smallest integer in  $\mathbb{N}_R$  larger than  $n_2$ , etc. In general, for each  $i \geq 1$ , let  $n_{i+1}$  be the smallest integer in  $\mathbb{N}_R$  larger than  $n_i$ . Define a map  $f : E_R \rightarrow E_R - \{a_{n_1}\}$  by  $f(a_{n_i}) = a_{n_{i+1}}$ . Then it is easy to see that  $f$  is a bijection, so  $E_R \subseteq R$  is infinite, so  $R$  is infinite. Contradiction. Thus,  $R \cup S$  is finite.

2. Let  $S$  be the smallest set such that there is an element  $e \in S$  and there is a bijection  $\varphi : S \rightarrow S - \{e\}$ . (“Smallest” here means that if  $T$  is another set with the same property and there is an injection from  $T$  to  $S$ , then there is a bijection from  $T$  to  $S$ . This corresponds to our intuition on finite sets, but covers both finite and infinite sets.)

Consider a set  $E = \{a_i \mid i \in \mathbb{N}\}$  defined as follows:  $a_1 = e$ ,  $a_{n+1} = \varphi(a_n)$  for any  $n \geq 1$  (in other words,  $a_n = \varphi^{n-1}(e)$ ). We claim that  $a_i \neq a_j$  if  $i \neq j$ . Indeed, without loss of generality (WLOG), suppose that  $i < j$ .  $i - j$  is a positive integer, so  $i - j \geq 1$ , so  $\varphi^{i-j}(e) = \varphi(\varphi^{i-j-1}(e)) \in \varphi(S) = S - \{e\}$ , hence  $\varphi^{i-j}(e) \neq e$ .  $\varphi$  is a bijection, so  $\varphi^{j-1}$  is also a bijection as a composition of  $j - 1$  bijections  $\varphi$ . Thus,  $\varphi^{i-j}(e) \neq e$  implies that  $\varphi^{j-1}(\varphi^{i-j}(e)) \neq \varphi^{j-1}(e)$ , i.e.  $\varphi^{i-1}(e) \neq \varphi^{j-1}(e)$ , that is  $a_i \neq a_j$ .

It is easy to see that  $f : E \rightarrow \mathbb{N}$  given by  $f(a_n) = n$  is a bijection. Hence,  $E \simeq \mathbb{N}$ . On the other hand,  $E$  is a subset of  $S$  and the restriction of  $\varphi$  to  $E$  is a bijection from  $E$  to  $E - \{e\}$ . However,  $S$  is the smallest set with such property, so there exists a bijection  $g : S \rightarrow E$ . But then  $f \circ g : S \rightarrow \mathbb{N}$  is also a bijection, so  $S \simeq \mathbb{N}$ .

3. (a) The “proof” implicitly assumes that we know there is an element  $b$  (same as or different from  $a$ ) such that  $a \sim b$ . However, the reflexivity property assumes no such thing, it simply states that  $a \sim a$  for all  $a$  with no additional assumptions.
- (b) We should simply make our implicit assumption explicit, so the new property should be: “For any element  $a \in S$ , if there exists an element  $b \in S$  such that  $a \sim b$ , then  $a \sim a$ .” In other words, if  $a$  is equivalent to any element, then it is equivalent to itself.