Math 301 **Assignment 1** **Due 02/01/02**

For fun: Do problems from Chapter 1, sections 1, 2, 3. These don't have to be turned in, but solving them will give you experience and insight you will need later. Feel free to ask me questions about these problems during my office hours.

Part A.

1. Let S be a set. We call S *infinite* if it possesses a non-trivial subset S_0 (i.e. $\emptyset \subsetneq S_0 \subsetneq S$) and there is a bijection (a 1-1 and onto function) $\varphi : S_0 \to S$. (So, $\mathbb{Z}$ and $\mathbb{N}$ are infinite sets.) We call S *finite* if S is not infinite.

   (a) Show the empty set, $\emptyset$, is finite. (takes ≤ 30 seconds)

   (b) If S is a set and $T \subseteq S$, show that if S is finite then T is finite.

2. A binary relation $\sim$ on a set S (i.e. one between two elements of S) is called an *equivalence relation* on S if it satisfies the following properties for all $a, b, c \in S$:

   Reflexivity: $a \sim a$.

   Symmetry: $a \sim b \implies b \sim a$.

   Transitivity: $(a \sim b \text{ and } b \sim c) \implies a \sim c$.

   Let S be a set, $a, b \in S$. Which of the following are equivalence relations, which are not, and why?

   (a) S is any set, $a \sim b$ if $a = b$.

   (b) S is any set, $a \sim b$ if $a \neq b$.

   (c) S is a set of integers, $a \sim b$ if $a \leq b$.

   (d) S is a set of integers, $a \sim b$ if $a - b$ is divides a fixed integer n.

   (e) S is a set of lines in the plane, $a \sim b$ if $a \parallel b$.

   (f) S is a set of lines in the plane, $a \sim b$ if $a \perp b$.

   (g) $S = \bigcup S_\alpha$ where all the S_α are mutually disjoint, nonempty sets (and index α is in some index set T), $a \sim b$ if both a and b are in the same S_α.

Part B.

1. For sets use the definitions of finite and infinite of Problem A1 above. Let R and S be finite sets. Prove that $R \cup S$ is a finite set.

   Remarks: You probably have an intuition that a finite set is one with, say, n elements. In that case, if R has n elements and S has m elements, then $R \cup S$ ought to have $\leq n + m$ elements. If you do wish to use that intuition here, you need to establish by strict argument that *your* notion of finite (i.e. having n elements) is logically the same as the notion of Problem A1. This is not so easy. Even if you do this, you are faced with the problem of $R \cup S$ and connection with addition in $\mathbb{N}$. At some point or points in your argument, some real ingenuity would be required.

2. Let S be the smallest set such that there is an element $e \in S$ and there is a bijection $\varphi : S \to S - \{e\}$. Show that $S \simeq \mathbb{N}$ (i.e. there is a bijection from S to $\mathbb{N}$).

   Remark: This proves that $\mathbb{N}$ is the smallest infinite set (up to a bijection).

3. (a) Consider the equivalence relation defined in Problem 2. What is wrong with the following proof that symmetry and transitivity imply reflexivity? "Let $a \sim b$, then $b \sim a$ by symmetry, so $a \sim a$ by transitivity (using $c = a$)."

   (b) Can you suggest an alternative of reflexivity property which will insure us that symmetry and transitivity do imply this alternative property?