Math 301A **Final Exam** **05/03/02** **Name**____________________________________

This exam is **due Tuesday, May 7, by 5pm, in my office, Carver 456**. However, if you have at least two other exams on Monday or at least two other exams on Tuesday, you may turn it in on Wednesday. You may consult the text for this course (or *Topics in Algebra* by I.N. Herstein), the book on reserve for this course, your notes taken in lecture, your homework, and sketches of solutions of homework problems. Do not use other books or papers or materials from a library or consult with any person other than myself. Answers to all questions except multiple choice require a rigorous proof. Show your argument *neatly.* Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the rules above. Remember to SHOW ALL WORK unless otherwise indicated. Each part of a problem is worth 5 points. The exam is worth 60 points; if you get more, the rest is extra credit.

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1. This problem consists of multiple-choice questions. Please *circle* the answer which is *always true* or *always correct*. Circle the *entire* answer, not just the letter labelling it. No work is required and no partial credit will be given for any part of this problem.

   (a) Consider the group G consisting of all 2×2 upper triangular real matrices $\sigma = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ for which $\det(\sigma) = 1$ and $a > 0$. Let G act on the the xy-plane (i.e. on $S = \mathbb{R}^2$) according to the rule:
   $$\sigma * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cy \end{pmatrix}$$
   Then the points (x, y) for which the stabilizer H is the subgroup of matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, all $b \in \mathbb{R}$, are:

      i. all points on the x-axis.
      ii. all points on the x-axis except $(0, 0)$.
      iii. all points on the x-axis except $(0, 0)$, $(1, 0)$, $(-1, 0)$.
      iv. all points in the right half-plane, i.e. $x > 0$.
      v. all points in the left half-plane, i.e. $x < 0$.

   (b) Given the same G and its action on $\mathbb{R}^2$ as in the previous problem:

      i. There are no points $(x, y) \in \mathbb{R}^2$ whose stabilizer is the identity subgroup of G.
      ii. There are points, but only finitely many, whose stabilizer is the identity subgroup of G.
      iii. Except for the points on a certain straight line, all other points have stabilizer the identity subgroup of G.
      iv. The points whose stabilizer is the identity subgroup of G form a straight line.
      v. None of the above is a true statement.

   (c) The group of rotational symmetries (reflectional symmetries are not included) of a regular tetrahedron is isomorphic to:

      i. S_4.
      ii. A_4.
      iii. V, the Klein's 4-group.
      iv. C_4, the cyclic group of order 4.
      v. None of the above.

   (d) The group U_{15} (see section 2.4) is isomorphic to:

      i. C_8.
      ii. D_4.
      iii. $C_4 \times C_2$.
      iv. $C_2 \times C_2 \times C_2$.
      v. None of the above.

2. Let $\phi : G \to G'$ be a homomorphism with $\ker \phi = K$, and let H be a subgroup of K. Determine $\phi^{-1}(\phi(H))$.

3. Show that $\mathbb{R}_\times = (\mathbb{R} - \{0\}, \times)$ and $\mathbb{R}_+ = (\mathbb{R}, +)$ are not isomorphic.

   Hint: Consider the kernel of any homomorphism from $\mathbb{R}_\times$ to $\mathbb{R}_+$.

4. Let m, n, a, b be integers such that $m, n > 0$ and $\gcd(m, n) = 1$. Prove that there is an integer x such that $x \equiv a \pmod{m}$ and $x \equiv b \pmod{n}$.

   Hint: This is a special case of the *Chinese Remainder Theorem*. First, show that such an integer exists when $a = 0$ and $b = 1$, then show it exists when $a = 0$ and b is any integer, and finally, show it exists for any integers a and b.

5. Let G be a finite group. Consider a p-Sylow subgroup H_p of G and a q-Sylow subgroup H_q of G, $q \neq p$. What is $H_p \cap H_q$?

6. Let G be a group of order p^2q, where p and q are prime, $q > p > 2$. Show that the q-Sylow subgroup of G is normal in G.

   Hint: Use all three Sylow's Theorems. Consider the number of q-Sylow subgroups of G. Determine the conditions it must satisfy and consider the possible choices.

7. (a) Construct a nonabelian group of order 21.

   (b) Show that all nonabelian groups of order 21 are isomorphic to each other. (In other words, there is only one such group up to an isomorphism.)

   Hint: Look at the proof of Theorem 2.8.5 and the hint in the problem 2.8.4. Consider all possible constructions (i.e. choices of generators and relations) and the ways to obtain them from one another.

8. Show that any normal subgroup of a group G is a disjoint union of some conjugacy classes of G.

9. Define the *cycle structure* of a permutation as the list of the lengths of its cycles in the cycle notation (with multiplicities) in non-increasing order. [E.g. $(12)(385)(49)$ has cycle structure $(3, 2, 2)$.] Show that any two permutations in S_n have the same cycle structure if and only if they are conjugate.

   Hint: This is not unlike what a change of basis does to matrices of linear transformations.

10. (a) Determine the sizes of the conjugacy classes of the elements of the alternating group A_5.

    Hint: First determine the possible cycle structures of the permutations in A_5. You can use problem 6, but remember that the conjugation in this problem is only by even permutations. Or, you may determine the size of the stabilizer of an element of each cycle structure, then find the size of each orbit. You may find the Class Equation useful as well. Or, you can also find all the conjugates directly, but remember that A_5 has $5!/2 = 120/2 = 60$ elements, so that may be somewhat time-consuming.

    (b) Show that A_5 is simple, i.e. has no nontrivial normal subgroups.

    Hint: Use Problems 8 and 10(a). Remember that each subgroup contains the identity element.

11. Let G be a finite group, and write $c(G)$ for the number of conjugacy classes of G. In general, this number will increase as $o(G)$ increases, so we introduce the number $\gamma(G) = c(G)/o(G)$. Clearly, $0 < \gamma(G) \leq 1$, and $\gamma(G) = 1$ if and only if G is abelian. Assume from now on that G is nonabelian.

(a) Prove that $\gamma(G) \leq 5/8$.

(b) If p is the smallest prime which divides $o(G)$, prove that

$$\gamma(G) \leq \frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^3}.$$

(c) Are the above bounds sharp? That is, can you find a group G with $\gamma(G) = 5/8$? similarly, for (b)?

Hint: Use the Class Equation. Consider the center of G and the rest of G.

12. Let G be a group of order pqr where $p < q < r$ are prime numbers.

(a) Show that either the q-Sylow subgroup of G or the r-Sylow subgroup of G is normal in G.

(b) Show that the r-Sylow subgroup of G is normal in G.