Math 314 **Homework 4** **Due 3/8/2007**

1. Determine (with proof) the edge-chromatic numbers χ' of the following graphs:

   (a) the wheel W_n for $n \geq 3$ (obtained by taking a cycle C_n and adjoining a new vertex called the *hub* and n edges called *spokes* from the hub to every vertex of C_n);

   (b) the double wheel D_n for $n \geq 3$ (obtained by taking a cycle C_n and adjoining two non-adjacent hubs with edges from each hub to every vertex of C_n) (*Hint:* note that there is a small special case here);

   (c) the dodecahedron graph D;

   (d) the n-cube Q_n. *Hint:* Assign a color to an edge based on the labels of its endpoints.

   Hint: You might try to use the "turning trick" on W_n and D_n.

2. Prove that if we color all edges of W_n for odd $n \geq 5$ with 2 colors, red and blue, then the resulting blue and red subgraphs cannot be isomorphic. *Hint:* It's all in the degree sequences. Consider the red degree sequence and the blue degree sequence, in particular, the larger of the red degree and the blue degree of the hub. If there were an isomorphism between red and blue graphs, where could the hub get mapped under such an isomorphism? What is the only choice for $n > 5$? Why is it impossible for odd n? You will see that $n = 5$ is a special case (why?). Do it carefully. Keep in mind that the red degree sequence and the blue degree sequence must add up element-wise to the degree sequence of the whole W_n.

3. Consider a graph G where the number of edges q is an integer multiple of the edge-chromatic number $\chi'(G)$. Show that there is a proper edge coloring of G where every color is used exactly q/χ' times. *Hint:* Choose the proper χ'-edge-coloring of G that has the least difference between the sizes of the smallest and largest color classes. If that difference is greater than zero, then look at the subgraph of G on the edges of two colors: one used the maximal number of times (say, red) and one used the minimal number of times (say, blue). What are the connected components of that red-and-blue graph and why? Which connected components *must* appear and why? Now try to recolor some of them properly so as to reduce the number of red edges and increase the number of blue edges. Derive a contradiction to your assumption (what was it?).

4. *(extra credit)* Call a graph G "randomly decomposable" into P_3 (a path on 2 edges) if no matter how you remove copies of P_3 from G you always use up all the edges of G (i.e. never get stuck until you have no edges left no matter how you do it).

   (a) Explain why C_4 is randomly decomposable into P_3 but C_6 is not.

   (b) Find a simple family of trees which are randomly decomposable into P_3.

   (c) Prove that the only connected graphs which are randomly decomposable into P_3 are C_4 and the family from (4b). *Hint:* a key step is to show that two of the trees on 4 edges are not randomly decomposable into P_3, so cannot be subgraphs of G.