Math 314 **Homework 3** **Due 3/1/2007**

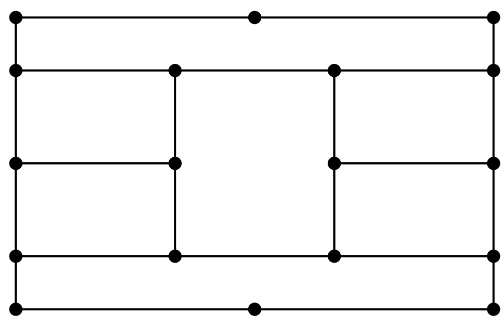
1. Exercise 1.1.24, page 17.

2. Determine (with proof) the chromatic number and diameter for the following graphs:

   (a) the dodecahedron graph D (the middle graph in Exercise 1.1.19 on page 16).

   (b) the n-cube Q_n. *Hint:* Color a vertex of Q_n based on its binary n-tuple label.

3. Consider the tennis graph below. Is it bipartite? Prove your answer. If no, what is the smallest number of edges you need to remove to obtain a bipartite graph? Prove your answer.



The tennis graph

4. Prove that if $\chi(G - v - w) = \chi(G) - 2$ for every pair of vertices v and w in a graph G, then G is complete. *Hint:* First prove that if G is not complete, then $\chi(G) \leq \chi(G - v - w) + 1$ for some pair of vertices v and w in G.

5. Prove that if every edge of a graph G lies on an odd number of cycles, then every vertex of G has even degree. *Hint:* Consider cycles through any given vertex v of G. Count, in two different ways, all pairs (e, C), where e an edge incident with v and C is a cycle through e. From the resulting equation, derive the desired conclusion.

6. The *product* graph $G \times H$ is defined as follows. Its vertex set $V(G \times H)$ is the Cartesian product $V(G) \times V(H)$, i.e. a set of ordered pairs (g, h) such that $g \in V(G)$ and $h \in V(H)$. Vertices (g_1, h_1) and (g_2, h_2) of $G \times H$ are adjacent if and only if either:

   - $g_1 = g_2$ and $h_1h_2 \in E(H)$, or
   - $h_1 = h_2$ and $g_1g_2 \in E(G)$.

   (a) Prove that $G \times H \cong H \times G$.

   (b) Prove that Q_n is isomorphic to $K_2 \times Q_{n-1}$ for $n \geq 1$.

   (c) Prove that $\chi(G \times H) \leq \chi(G)\chi(H)$. *Hint:* Color each vertex (g, h) in $G \times H$ with one of $\chi(G)\chi(H)$ colors based on the colors of the vertex g in G and the vertex h in H so as to obtain a proper coloring.

   (d) *(extra credit)* Prove that in fact $\chi(G \times H) = \max(\chi(G), \chi(H))$. *Hint:* For colors, use remainders modulo $\max(\chi(G), \chi(H))$. As before, assign a color to (g, h) in $G \times H$ based on the color of g in G and the color of h in H, but in a different, more economical way that still yields a proper coloring.