Math 314 **Sketch of Solutions to Homework 2** **Spring 2007**

1. There are 11 non-isomorphic trees on 7 vertices. To see this, classify the trees according the number of leaves. Then classify them further according to their degree sequence. Any tree on 7 vertices has $7 - 1 = 6$ edges, so the sum of all degrees of its vertices must be $2 \cdot (7 - 1) = 12$. Any tree must also have at least two leaves, i.e. two vertices of degree 1. Note that nonleaves in a connected graph such as a tree have degrees ≥ 2.

Case 1: 6 leaves and 1 nonleaf with degree $12 - 6 = 6$. Hence, the degree sequence is $6, 1, 1, 1, 1, 1, 1$, which yields only the *star* tree $S_6 = K_{1,6}$.

Case 2: 5 leaves and 2 nonleaves whose degrees sum to $12 - 5 = 7$. This yields two degree sequences: $5, 2, 1, 1, 1, 1, 1$ and $4, 3, 1, 1, 1, 1, 1$. Each of these sequences yields a unique tree.

Case 3: 4 leaves and 3 nonleaves whose degrees sum to $12 - 4 = 8$. This yields degree sequences $4, 2, 2, 1, 1, 1, 1$ and $3, 3, 2, 1, 1, 1, 1$. The sequence $4, 2, 2, 1, 1, 1, 1$ yields two non-isomorphic trees: the vertex of degree 4 is adjacent to a leaf in one and non-adjacent to leaves in the other. The sequence $3, 3, 2, 1, 1, 1, 1$ also yields two non-isomorphic trees: the two vertices of degree 3 are adjacent in one and non-adjacent in the other.

Case 4: 3 leaves and 4 nonleaves whose degrees sum to $12 - 3 = 9$. This yields a unique degree sequence: $3, 2, 2, 2, 1, 1, 1$, which yields three non-isomorphic trees: the vertex of degree 3 is adjacent to 2 leaves in the first tree, 1 leaf in the second tree, and no leaf in the third tree.

Case 5: 2 leaves and 5 nonleaves whose degrees sum to $12 - 2 = 10$. This yields a unique degree sequence: $2, 2, 2, 2, 2, 1, 1$, which yields the path P_6 (6 edges).

2. (a) Any graph must have an even number of odd vertices. Since all vertices in G have odd degrees, G must have an even number p of vertices. Since G is a tree, the number of edges in G, $q = p - 1$, hence q is odd.

   (b) K_4 has all 4 vertices of degree 3 and $(3 \cdot 4)/2 = 6$ edges.

3. We will show that if every vertex in a graph G has degree at least $(p - 1)/2$, then any two vertices are either adjacent or have a common neighbor, hence G is connected. Suppose neither is true for some vertices u and v of G. Then the number of vertices in G is at least $1 + (p - 1)/2 + 1 + (p - 1)/2 = p + 1$ (u and its neighbors plus v and its neighbors). But $p + 1 > p$, hence we have a contradiction, so our assertion above is true.

   This bound is sharp. A graph $\overline{K_2}$ has $p = 2$ vertices, each of degree $(p - 2)/2 = 0$, yet it is disconnected.

4. (a) $K_{3,3}$ (why?). (b) Three copies of K_3 joined at one vertex (why?).