Math 314 **Final Exam** **Spring 2007**

This exam is due Thursday, May 3, in my office (Carver 456) or in my email. You may consult the text, handouts and homework solutions for this course, your notes taken in lecture and your homework. Do not use any other books or papers or materials from a library or consult with any person other than myself. Please sign your name on your completed work and write, just above your signature, a statement to the effect that you have observed the above rules. SHOW ALL WORK except for Problem 1. Mere oracular answers will get little, if any, credit.

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1. Complete the table by finding all nonnegative integers m and n for which the graphs below have the following properties:

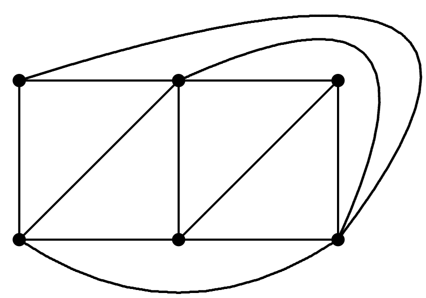
	C_n	$K_{m,n}$	K_n	W_n	D_n	Q_n
planar						
maximal planar						
maximal bipartite planar						

2. Color the edges of $K_{4,4}$ with 2 colors so that there is no monochromatic $K_{2,2}$. *Hint:* Find a nice, symmetric decomposition.

3. Prove that $\mathrm{cr}(K_6) = 3$.

4. Find the labeled tree associated by the Prüfer method with the sequence $(2, 2, 5, 5, 5, 8, 8)$.

5. Find another plane drawing of the graph below where all edges are straight lines. (Label the vertices on both drawings so that the isomorphism is apparent.) (*extra credit* for a nice, symmetric drawing)



6. Connect A to a, B to b, C to c, and D to d in Figure 1 by edges that are inside the rectangle and do not cross each other.

7. Prove that the degree sequence $(n, n, n-1, n-1, , n-2, n-2, \ldots, 2, 2, 1, 1)$ is graphic. *Hint:* Construct a graph with this degree sequence as follows. Take $2n$ vertices $u_1, \ldots, u_n; v_1, \ldots, v_n$ and decide whether or not to join each pair u_i and v_j by an edge based on whether or not the pair of indices i and j satisfies a certain simple condition to be discovered by you. Then prove that, for every $i = 1, 2, \ldots, n$, each of u_i and v_i is adjacent to exactly i vertices in the graph you constructed.

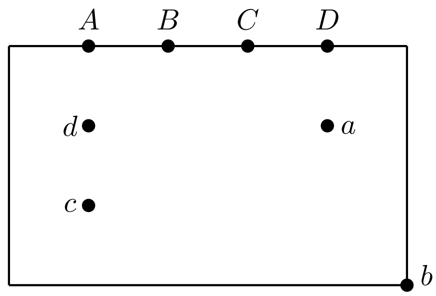


Figure 1: Dudeney's Puzzle

8. A bipartite graph is called *balanced* if every 2-coloring of the vertices uses every color the same number of times.

   (a) Prove that a regular bipartite graph of positive degree must be balanced. *Hint:* Take any red-and-blue vertex coloring of the graph. Count the edges as starting at the red vertices and as starting at the blue vertices. *Caution:* This bipartite graph does not have to be complete bipartite.

   (b) Prove that a Hamiltonian bipartite graph on ≥ 2 vertices must be balanced. *Hint:* Take any red-and-blue coloring of the vertices and consider the Hamilton cycle.

9. For an integer $n \geq 2$, the graph M_n is defined by taking a cycle C_{2n} on $2n$ vertices $1, 2, \ldots, 2n$ and joining each pair of opposite vertices (j and $n + j$ for $j = 1, 2, \ldots, n$) on the cycle. (So M_n is 3-regular.)

   (a) Find $\chi(M_n)$. Prove your answer. *Hint:* The answer depends on whether n is odd or even. There is also a small special case.

   (b) Find $\chi'(M_n)$. Prove your answer.

   (c) Prove that M_n is not planar for $n \geq 3$. *Hint:* First consider M_3, determine which known graph M_3 is isomorphic to, then use Kuratowski's theorem on M_n.

   (d) Prove that the crossing number $\text{cr}(M_n) = 1$ for $n \geq 3$. *Hint:* Recall that $\text{cr}(M_n)$ is the least number of crossings a simple drawing of M_n in the plane must have. For this, redraw M_n in a completely different way by uncrossing as many edges as possible.

10. (a) Prove that if n is odd, then D_n is not decomposable into P_7. *Hint:* Count edges.

    (b) Prove that if $n \geq 6$ is even, then D_n is decomposable into P_7. *Hint:* Find an initial P_7 and use the turning method.

    (c) Prove that D_n is never decomposable into P_8. *Hint:* Counting argument along the following lines. How many edges do D_n and P_8 have? How many copies of P_8 would D_n have to contain? How many edges on each P_8 can be adjacent to a given hub? On all P_8's together? Find an upper bound and derive a contradiction.

11. (*extra credit*) Prove that a regular graph which is decomposable into spanning trees must be complete. *Hint:* It's all in the numbers. Let G be an r-regular graph with p vertices and q edges, which is decomposable into k spanning trees. Express q in two ways using these parameters. Use a divisibility argument to prove that $r = p - 1$.